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"The Third Elliptic Integral and the Ellipsotomic Problem." By A. G. GREENHILL, F.R.S. Received December 22, 1903,—Read January 21, 1904.

(Abstract.)

The elliptic integral of the third kind, which makes its appearance in a dynamical problem, is of the circular form in Legendre's classification, and thus the Jacobian parameter is a fraction of the imaginary period, so that the expression by means of the theta function can no longer be considered as reducing the variable elements from three to two.

Burkhardt* has given a series rapidly convergent for the numerical calculation of such cases; but the object of this memoir is to develop the exact expression by means of an idea of Abel, given in the first volume of 'Crelle's Journal,' 1826, "Ueber die Integration der Differential-Formel

(1) $$\rho dx / \sqrt{\mathrm{R}},$$

wenn R and ρ ganze Functionen sind."

Abel proves practically that when the elliptic parameter is an aliquot μth part of a period, the third elliptic integral and the associated theta functions depend on the μth root of an algebraical function.

Thus, as shown in the paper, if we take the Jacobian elliptic parameter

(2) $$v = 2\mathrm{K}'i/\mu,\ \mu = 2n+1, \text{ an odd integer},$$

the third elliptic integral in the form

(3) $$\mathrm{I}(v) = \int \frac{\mathrm{P}(v)\, t^2 - \frac{1}{2}xy}{t^2} \frac{dt^2}{\sqrt{(\mathrm{T}_1\mathrm{T}_2)}},$$

* 'Elliptic Functions,' § 126.

subject to the condition

$$(4) \qquad \gamma_\mu = 0$$

is such that

$$(5) \qquad \begin{aligned} &2t^{n+\frac{1}{2}} \exp. (n+\tfrac{1}{2})\, \mathrm{I}(v)\, i \\ &= (t^{n-1}+h_1 t^{n-2}+h_2 t^{n-3}+ \ \ldots\ )\sqrt{\mathrm{T}_1} \\ &+i\,(t^{n-1}-h_1 t^{n-2}+h_2 t^{n-3}- \ \ldots\ )\sqrt{\mathrm{T}_2}, \end{aligned}$$

where

$$(6) \qquad \mathrm{T}_1,\ \mathrm{T}_2 = 2t^3 \pm (1+y)\,t^2+2xt \pm xy,$$

and x, y, γ_n are the functions employed by Halphen.*

The calculation of the coefficients h_1, h_2, . . . can be carried out by the method of *réduites*,† avoiding the continued fractions employed by Abel, which make the order of the result higher than is necessary.

Here P (v) is of the nature of a zeta function, and equations are given for its calculation, as an algebraical function of a parameter.

But in most dynamical problems, such as Poinsot's herpolhode and the associated motion of the symmetrical top, the Jacobian parameter is of the form

$$(7) \qquad v = \mathrm{K}+2\mathrm{K}'i/\mu,$$

and this requires the resolution of T_1 and T_2, equivalent to a change to an even value, $4n+2$, of μ.

Then if ρ, ϖ denote polar co-ordinates in the invariable plane of a Poinsot herpolhode, or of the angular momentum vector of a top, and t' denotes the time, we can take

$$(8) \qquad pt' - \varpi = \mathrm{I}(v), \qquad \mathrm{M}\rho/k = t,$$

so that in such a curve

$$(9) \qquad 2\,(\mathrm{M}\rho/k)^{n+\frac{1}{2}} \exp. (n+\tfrac{1}{2})\,(pt'-\varpi)\, i$$

is an algebraical function; and the curve is purely algebraical when the constants are so chosen as to cancel the secular term pt'.

The projection of the path of the centre of gravity of the top is the hodograph of the herpolhode of angular momentum, and is thus obtainable by a differentiation of the above; some of these applications are developed in a memoir on the top, now appearing in the 'Annals of Mathematics.'

With $\mu = 8n+4$ or $8n$ a further reduction is possible in degree. We find for $\mu = 8n+4$,

$$(10) \qquad \begin{aligned} &(\mathrm{D}-x^2)^{n+\frac{1}{2}} \exp. (2n+1)\, \mathrm{I}(v)\, i \\ &= [\mathrm{R}_0+\mathrm{R}_1 x+ \ \ldots\ +(-1)^n x^{2n}]\sqrt{\tfrac{1}{2}\mathrm{X}_1} \\ &+i\,[\mathrm{R}_0-\mathrm{R}_1 x+ \ \ldots\ +(-1)^n x^{2n}]\sqrt{\tfrac{1}{2}\mathrm{X}_2}. \end{aligned}$$

* Halphen, 'Fonctions Elliptiques,' vol. 1, p. 102.

† 'Fonctions Elliptiques,' vol. 2, p. 576.

$$(11) \qquad X_1, X_2 = 1 \pm (o^{-1} - o)\, x - x^2,$$

where o denotes the octahedron-irrationality, $o = \sqrt{\kappa}$.

But with $\mu = 8n$, this changes to

$$(12) \qquad (D - x^2)^n \exp. 2n\mathrm{I}(v)\, i$$
$$= [R_0 + R_1 x + \; . \; . \; . \; + (-1)^n x^{2n-1}] \sqrt{\tfrac{1}{2}X_1}$$
$$+ i\,[R_0 - R_1 x + \; . \; . \; . \; - (-1)^n x^{2n-1}] \sqrt{\tfrac{1}{2}X_2}$$

$$(13) \qquad X_1, X_2 = 1 \pm (o^{-1} + o)\, x + x^2.$$

The results are worked out in the memoir for numerical values as far as possible, starting with the simplest, 3, 4, 5 . . . , and the application is shown to other associated mechanical problems, such as central orbits, the spherical catenary, the elastica and velarium.

Provided with a list of these integrals, the student of Applied Mathematics will be able to effect the complete discussion of many mechanical problems now abandoned in an unfinished state; at the same time the exploration along the simplest line of progress is effected of the general analytical field, and mathematical research is guided along a road likely to lead to useful development in the theory of elliptic functions.

Incidentally the elliptic section or division values (Theilwerthe) are determined, as well as those of the zeta and theta functions, as algebraical functions of a parameter, in a form such that

$$(14) \qquad \left(\frac{\Theta 2\mathrm{K}/\mu}{\Theta 0}\right)^{\mu} \text{ and } \left(\frac{\mathrm{H}2\mathrm{K}/\mu}{\mathrm{HK}}\right)^{\mu}$$

are of simple algebraical character; and it is shown that this, the Ellipsotomic Problem, as it may be called by analogy, depends essentially on the discussion of the curve given by (4), which may be called the ellipsotomic equation and curve in the co-ordinates x and y, or on that of a reduced form, involving the determination of its class, and the expression of its co-ordinates as functions of a parameter.

The coefficients in the Transformation of elliptic functions are symmetric functions of these section values, so that the Transformation may be considered as determined incidentally, but as the Transformation theory has no utility in dynamical applications, this branch of pure analysis has not been pursued.